Non-Newtonian Stability

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ORR-SOMMERFELD EQUATION

Consider the incompressible Navier-Stokes equations for a Newtonian Fluid with constant density.

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla P}{\rho} + \nu\nabla^2\mathbf{u} \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{2}$$

We will assume the some perturbation expansion of the form

$$\mathbf{u} = (U(z), 0) + \epsilon(u_1, v_1) + \dots$$

We then define a streamfunction as follows:

$$\psi = \exp(\lambda t)\phi(k, y)$$

Collecting terms we arrive at the Orr-Sommerfeld equation

$$\nu D^2 D^2\psi - ikUD^2\psi + ik\left(\partial_z^2 U\right)\psi = \lambda D^2\psi$$

MODIFIED ORR-SOMMERFELD EQUATION

Now, what if we have a non-Newtonian Fluid? The momentum equations change, so we are forced to write

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla P}{\rho} + \nabla\cdot\left(\nu(I_2)e_{ij}\right) \tag{3}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{4}$$

where I_2 is a second invariant of the strain tensor. We will assume the same boundary as before.

Viscosity Equation Of course now, the viscous term no longer has a nice form. In this case we will assume a typical power law relation

$$\nu = \nu_0\left(2e_{ij}e^{ij}\right)^{\frac{n-1}{2}}$$

O(1) Solution

As before, we will assume the the O(1) solution will be one-dimensional. Thus, assuming a static solution with a pressure gradient, symmetric about the the centre, we find that

$$U(z) = \left(\frac{2}{\nu_0}\right)^{\frac{1}{n}}\frac{n}{n+1}\left(1 - |z|^{\frac{n+1}{n}}\right)$$

Under some particular value of $\partial_x P$. Notice now that for a Newtonian fluid, n=1, and the solution reduces to the previous result.

O(ϵ) Solution If we define our streamfunction as before, we can again derive an eigenvalue relation

$$\left[\frac{1}{\mathrm{Re}}\left(C_0\partial_z^4 + C_1\partial_z^3 + C_2\partial_z^3 + C_3\partial_z + C_4\right) + kU(z)\left(\partial_{zz} - k^2\right) - k\partial_{zz}U(z)\right]\psi = \lambda\left[\partial_z^2 - k^2\right]\psi \tag{5}$$

Where we define

$$C_0 = n\left(\partial_z U(z)\right)^{n-1}$$
$$C_1 = 2(n-1)n\left(\partial_z U(z)\right)^{n-2}\partial_{zz}U(z)$$
$$C_2 = 2k^2(n-2)\left(\partial_z U(z)\right)^{n-1} + \partial_z\left[n(n-1)\left(\partial_z U(z)\right)^{n-2}\partial_{zz}U(z)\right]$$
$$C_3 = 2k^2(n-1)(n-2)\left(\partial_z U(z)\right)^{n-2}\partial_{zz}U(z)$$
$$C_4 = n\left(\left(\partial_z U(z)\right)^{n-1}k^4 + k^2\partial_z\left[(n-1)\left(\partial_z U(z)\right)^{n-2}\partial_{zz}U(z)\right]\right)$$

Notice now, that if n=1, the problem again reduces to the Orr-Sommerfeld equation derived previously

PSEUDOSPECTRA

If λ is an eigenvalue of matrix A then it will be true that

$$\|A - \lambda I\| = 0$$

Trefethen expands this idea by looking to the pseudospectra of A, defined $\Lambda_\epsilon(A)$ as

$$\Lambda_\epsilon(A) = \{z \in \mathbb{C} : \sigma_{min}(zI - A) \le \epsilon\}$$

The hope is that the pseudospectra will help us better understand the stability of the operator.

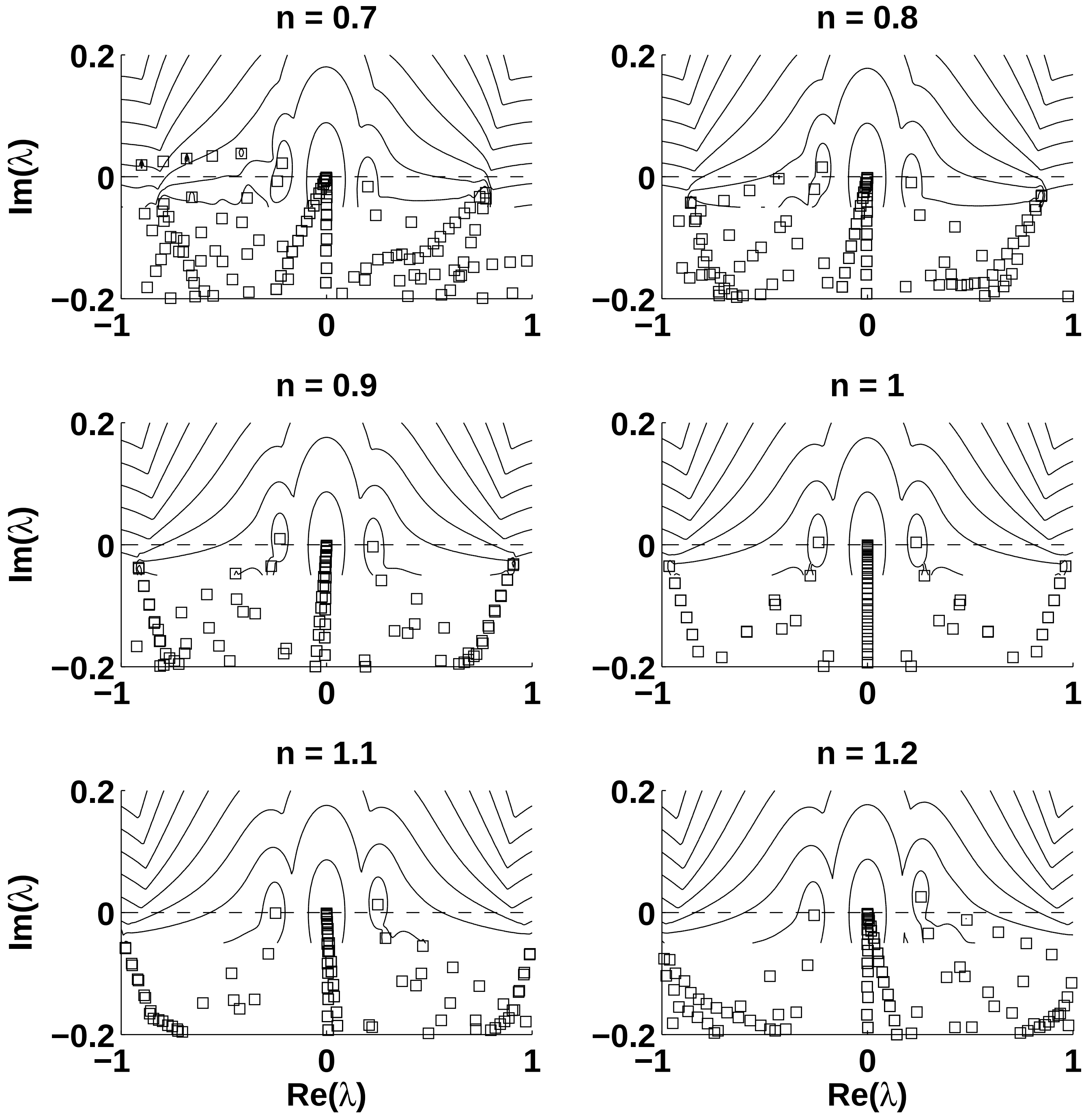

FIG. 1: Pseudospectra of the modified Orr-Sommerfeld equation for varying values of n. Re = 10,000. N = 128.