Introduction and Motivation

In recent years there has been a push to better understand stratified turbulence, looking towards modelling the ocean and atmosphere. The purpose of this work is to investigate linear stability of a vortex pair as precusor to understanding the transition to stratified turbulence.

In 2000, Billant and Chomaz (BC) performed an experiment in which they created a propagating counter rotating vortex pair (resembling Lamb-Chaplygin dipole) aligned vertically in a tank of stratified water. They noticed is that initially the vortex pair behaved similiar to the non-stratified case. However, as time went on, the pair evolution exhibited a new instability not seen in the non-stratified case. The instabilities of the vortex pair in a non-stratified fluid are well understood due to aeronautical applications (Crow and elliptic instabilities) but this new instability had no theoretical explanation. BC named this new instability the "zig-zag" instability because the fluid had a zig-zag like structure.

BC provided a theoretical and numerical explanation of this new zig-zag instability. By doing linear stability analysis of the Boussinesq equations, they were able to perturbatively and numerically verify and predict the existence of this zig-zag instability. Through these two methods, they were able to predict the mode (wavenumber) with the largest growth rate, which agreed with the experimental results, to within experimental error.

However BC only explore the long wavelength regime growth rates in their numerical solutions. Our project is looking at what happens if we probe shorter wavelengths. To skip right to the punchline, what we have found is that there exist shorter wavelength modes that have equivalent or even larger growth rates then those of the zig-zag instability.

Equations and Numerics

We now proceed with a linear stability analysis of a Lamb-Champlygin dipole, a 2D solution to the Euler equations, with radius R and velocity U. The governing equations are the Boussinesq equations along with the density equation

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0}\nabla P - g\frac{\rho'}{\rho_0}e_z + \nu\nabla\mathbf{u}, \qquad \frac{D\rho'}{Dt} + u_z\frac{\partial\bar{\rho}}{\partial z} = D\nabla\rho', \qquad \nabla\cdot\mathbf{u} = 0$$

where the mean density profile $\bar{\rho}(z)$ defines the buoyancy frequency $N^2 = -g/\rho_0\partial\bar{\rho}/\partial z$, assumed to be constant here. We can non-dimensionalise this equation by taking the characteristic velocity to be U, length L, time scale R/U, pressure $\rho_0 U^2$, and density $\rho_0 U^2/(gR)$. This allows us to define $Re = UR/\nu, Sc = \nu/D, F_h = U/RN$. We now linearise about the 2D basic state

$$\mathbf{u} = \mathbf{u}_{0h} + \tilde{\mathbf{u}}, \qquad P = P_0 + \tilde{P}, \qquad \rho' = \rho_0 + \tilde{\rho}'.$$

Grinding through the algebra and doing the non-dimensionalisation we obtain

$$\frac{\partial\tilde{\mathbf{u}}}{\partial t} + \omega_{z0}e_z\times\tilde{\mathbf{u}} + \tilde{\boldsymbol{\omega}}\times\mathbf{u}_{h0} = -\nabla(\tilde{P} + \mathbf{u}_{h0}\cdot\tilde{u}) - \tilde{\rho}'e_z + \frac{1}{Re}\nabla^2\tilde{u},$$
$$\nabla\cdot\tilde{\mathbf{u}} = 0, \qquad \frac{\partial\tilde{\rho}'}{\partial t} + \mathbf{u}_{h0}\cdot\nabla_h\tilde{\rho}' - \frac{1}{F_h^2}\tilde{u}_z = \frac{1}{ScRe}\nabla^2\tilde{\rho}'$$

Because the dipole is aligned along the z direction we can express the terms as a normal mode

$$\tilde{\mathbf{u}} = \mathbf{u}'(x,y,t)e^{ik_z z} + \text{c.c.}$$

From here can now take the 2D Fourier transform and define a projection operator to eliminate pressure to obtain a set of equations in Fourier space.

$$\frac{\partial\hat{u}}{\partial t} = P(k)[\mathbf{u}\widehat{\times\,\omega_{z0}}e_z + \widehat{\mathbf{u}_{h0}\times}\,\omega - \hat{\rho}'e_z] - \frac{k^2}{Re}\hat{u},$$
$$\frac{\partial\hat{\rho}'}{\partial t} = -ik\widehat{\mathbf{u}_{h0}\rho'} + \frac{1}{Fh^2}\hat{w} - \frac{k^2}{ScRe}\hat{\rho}'.$$

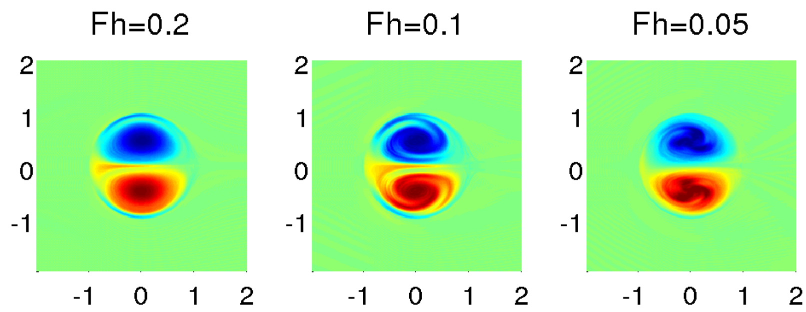

Figure 1: Plot of the vertical vorticity of the full perturbed state (base state plus perturbation) for Re= 10000 at the second peak growth rate for various Froude.

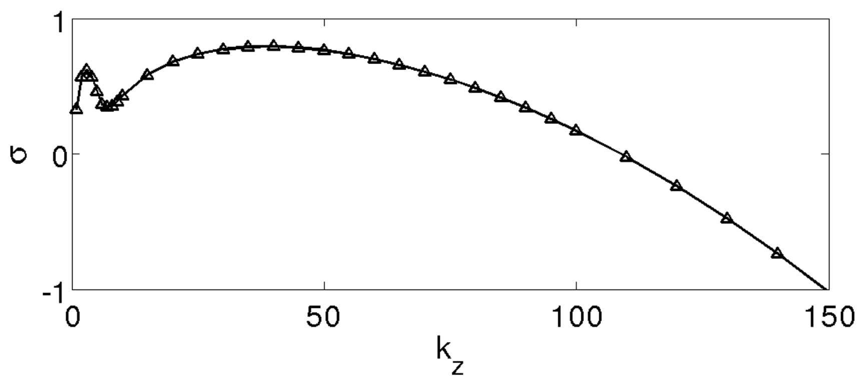

Figure 2: Plot of the growth rate, σ verus vertical wavneumber k_z for $F_h = 0.2, Re = 10000$.

To find the maximum growth rate, we integrate the above equations for a very long time and it can be shown that the leading eigenmode behaves as

$$\lim_{t\to\infty} \mathbf{u}(x,y,t) = \mathbf{U}(x,y)e^{\sigma t}$$

where $\mathbf{U}(x,y)$ is the leading eigenmode and σ is the leading eigenvalue. This σ is then calculated by integrating over the total kinetic energy.

To solve we use a spectral method with 2/3 dealiasing, 2nd order Adams-Bashforth time-stepping scheme. We have found that a grid size of 512×512 gives robust results for Re<= 20000.

We have done a parameter sweep looking at $F_h = 0.2, 0.1, 0.05, 0.025, Re = 20000, 10000, 5000, 2000$ for vertical wavenumbers $k_z = 1, .., 160$.

Results and the future

See plots for some numerical results for Re=10000, Fh= $0.2, 0.1, 0.05$.

The first peak in the growth rate curve is the zig-zag instability explained by BC. As can be seen in this case, there exists a second peak which is comparable to the first peak. So what is the cause of this second peak and potentially another oscillatory instability? As of today, unexplained. We are currently processing the data and trying to get a better feel for what is going on. Currently we are looking at vortex form of the linear equations and a different nondimensionalisation argument of the Boussinesq equations to determine which terms are contributing to this second peak.

In the future there are plans to go to a full DNS of the Boussinesq equations.