Embedded method for solving the DJL equation

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Given a constant background velocity, U_0, the steady state Euler equations of motion for an incompressible, Boussinesq fluid can be simplified to a single equation for the isopycnal displacement, $\eta(x, z)$. This equation is called the Dubreil-Jacotin-Long (DJL) equation, and with boundary conditions and bottom topography $h(x)$, is given by

$$\nabla^2\eta + \frac{N^2(z-\eta)}{U_0^2}\eta = 0 \tag{1}$$

$$\eta \to 0 \text{ as } x \to \pm\infty \tag{2}$$

$$\eta = 0 \text{ at } z = 1 \text{ and } \eta = h(x) \text{ at } z = h(x), \tag{3}$$

where

$$N^2(z) = -\frac{g}{\rho_0}\bar{\rho}'(z).$$

In these definitions, ρ_0 is a constant reference density, $\bar{\rho}(z)$ is the density profile far upstream of the topography, and g is the acceleration due to gravity. The topography $h(x)$ is isolated, which means that it tends to zero as $x \to \pm\infty$.

This equation can yield very large trapped disturbances when U_0 is close to the conjugate flow speed, c_j. We consider supercritical flows, $U_0/c_j > 1$, for which waves cannot propagate upstream. In this case, the steady DJL theory can yield very large, trapped waves for certain density profiles and background velocities close to c_j. These waves are described in the Figure.

The density profile we have used is

$$\bar{\rho}(z) = 1 - \Delta\rho \tanh\left(\frac{z - z_j}{d_j}\right),$$

where $\Delta\rho = 0.02$ and $d_j = 0.1$. We have considered several values of z_j: $z_j = 0.6$ and $z_j = 0.4$. Large waves can occur for $z_j = 0.6$ for hole topography and $z_j = 0.4$ for hill topography.

We have solved the DJL equation with two methods: 1) a mapped method which involves a mapping relation between the computational domain and physical domain and 2) an iterative method where the DJL equation is solved on a rectangular domain and the bottom boundary condition is implemented iteratively. With the mapped method, the bottom boundary condition is easily set using a Dirichlet condition. However, this method is expensive since it results in large, full matrices due to the mapping. We have found that the iterative method is computationally more efficient and can implement the bottom boundary condition with satisfactory error (O(10^{-6})).

The bottom boundary condition is implemented using a method by Laprise and Peltier:

$$\eta^{N+1}(z=0) = \eta^N(z=0) - err^N(z=h)$$
$$err^N(z=h) = \eta^N(z=h) - h.$$

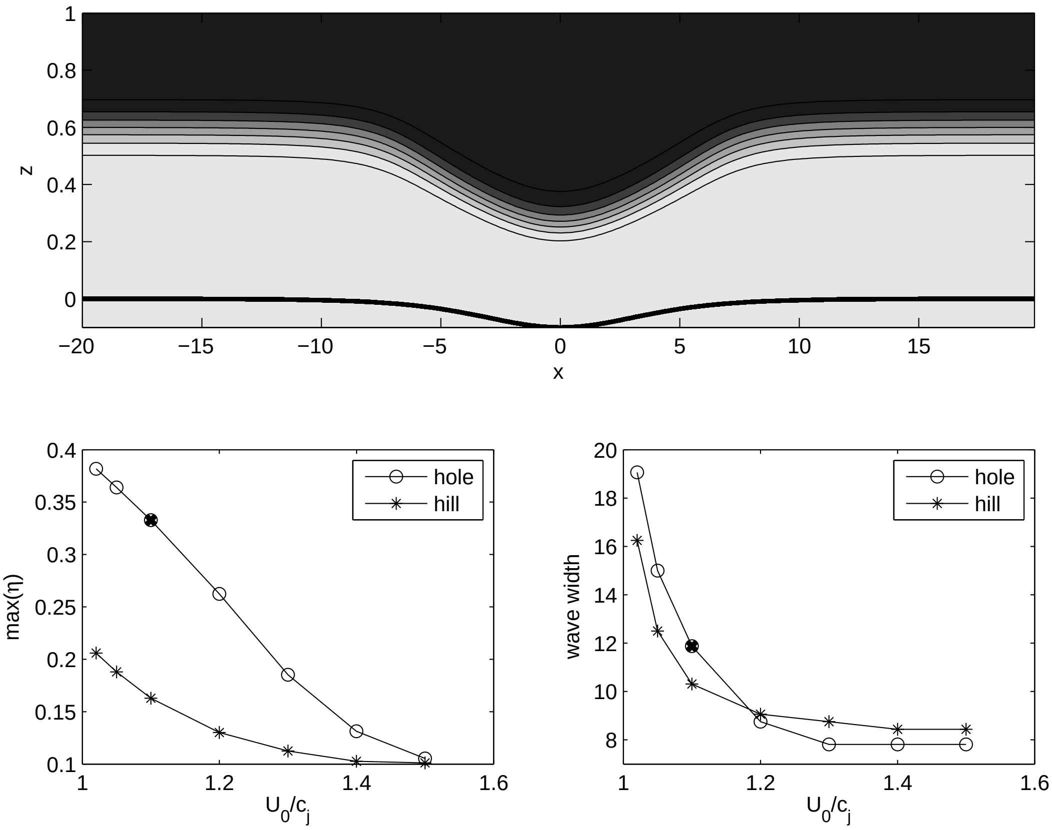

Figure 1: Density contours for a large trapped wave over hole topography with z_j=0.6 (top). Maximum wave amplitude (bottom left) and a measurement of wave width (bottom right) for several background speeds and hill/hole topography. The wave width is defined as the location where the perturbation surface velocity becomes half of its extreme value. The hill cases take $z_j = 0.4$ and the hole cases take $z_j = 0.6$. The shaded circles correspond to the wave in the upper panel.