Dealing with insignificant fast waves in numerical models:

Kinematic constraints vs. implicit time-stepping

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I. Introduction

Recall that the oceanic shallow water equations over a flat bottom boundary at $z = -H$ are:

$$\begin{aligned} (h\mathbf{u})_t + \nabla \cdot (h\mathbf{u} \otimes \mathbf{u}) &= -gh\nabla h + fh\mathbf{u} \times \mathbf{k} , \\ h_t + \nabla \cdot (h\mathbf{u}) &= 0 , \end{aligned} \tag{1}$$

where $h(x, y, t) = H + \eta(x, y, t)$ is the water column height, $\mathbf{u} = (u(x, y, t), v(x, y, t))$ is the depth-averaged velocity, $g = 9.81$ ms^{-2}, and f is the Coriolis frequency.

At high-enough latitudes and at long enough length scales (i.e., small Rossby number), we may wish to neglect the surface gravity waves since they are too fast[1] to be important for the length and time scales of interest. That is, we expect the dynamics to be dominated by nonlinear advection and planetary (i.e., Rossby) waves only.

We can "filter out" the gravity waves by making the rigid-lid approximation $\eta \approx 0$, hence $h \approx H = const$. After some light algebra and being careful with the depth-integrated pressure terms we find

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} &= -\nabla p + f\mathbf{u} \times \mathbf{k} , \\ \nabla \cdot \mathbf{u} &= 0 , \end{aligned} \tag{2}$$

the homogenous incompressible Euler equations. Here, p is the rigid-lid pressure (scaled by a reference density, ρ_0). If desired, the equations can be further simplified/reduced by deriving the vorticity equation for $\zeta = (\nabla \times \mathbf{u}) \cdot \mathbf{k}$ and introducing a streamfunction ψ such that $\mathbf{u} = \nabla \times (\psi \mathbf{k})$.

II. Methods

From a numerical stand-point both the full shallow water system (1) and the filtered equations (2) offer their advantages and disadvantages.

Set (1) is a strictly hyperbolic system of conservation laws, making it readily solvable by finite volume and discontinuous[2] Galerkin finite element methods. On the other hand, the presence of gravity waves means that, for stability, explicit time-stepping formulas must necessarily have time-steps smaller than $\Delta x / \sqrt{gH}$, where Δx is the grid-spacing. This restriction can be prohibitive for the scales of interest, and implicit time-stepping should be explored instead.

Set (2) circumvents the time-step issue of set (1), but in the end a price must be paid. Finite volume and finite element methods typically require staggered grids or unequal order velocity-pressure approximations to achieve stability while enforcing $\nabla \cdot \mathbf{u} = 0$. Furthermore, the equations are not a hyperbolic system, so methods that only weakly enforce continuity can no longer rely on the theory of hyperbolic conservation laws to exchange information between grid cells.

III. Time-stepping the filtered equations

Talk of spatial discretization methods aside, let's see how one would time-step set (2). The time-step imposed by the nonlinear advection and Coriolis terms shouldn't be too small, so those can be stepped explicitly. Since we don't have an evolution equation for p we'll have to solve for it implicitly. The scheme looks like,

$$\mathbf{u}^{n+1} + \beta_0 \Delta t \nabla p^{n+1} = \mathbf{u}^n + \Delta t N(\mathbf{u}^n) , \tag{3}$$

$$\nabla \cdot \mathbf{u}^{n+1} = 0 . \tag{4}$$

Here N represents an explicit time-stepping operator, and $0 < \beta_0 \leq 1$ depends on which implicit time-stepping formula is used. Effectively, we're solving for the p that will give us $\nabla \cdot \mathbf{u} = 0$ (conserves mass) at the new time-step.

At this point, an algebraic reduction (think row-reduction) is often applied to de-couple the equations, thereby reducing the size of the linear system to be solved. The reduction is done by first taking $\nabla \cdot$ (3), and substituting (4). The problem becomes:

$$\nabla^2 p^{n+1} = \frac{1}{\beta_0 \Delta t} \nabla \cdot \mathbf{u}^* , \tag{5}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \beta_0 \Delta t \nabla p^{n+1} . \tag{6}$$

where $\mathbf{u}^*$ is the RHS of (3) (called the "predicted velocity").

IV. Time-stepping the hyperbolic equations

After recognizing that the pressure gradient and divergence terms in (1) are responsible for the propagation of the fast gravity waves, we might try a first order implicit backward

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[1] A *back of the envelope* calculation shows $c_0 = \sqrt{gH} \sim 200$ ms^{-1}, where $H = 4$ km is a rough average ocean depth.

[2] Read: weakly continuous.

Euler (BE) scheme ($\beta_0 = 1$) on them:

$$(h\mathbf{u})^{n+1} \quad + \quad \Delta t g h^n \nabla h^{n+1} = (h\mathbf{u})^* \,, \qquad (7)$$

$$(h)^{n+1} \quad + \quad \Delta t \nabla \cdot (h\mathbf{u})^{n+1} = h^n \,, \qquad (8)$$

where $(h\mathbf{u})^* = (h\mathbf{u})^n + \Delta t N(h\mathbf{u}^n)$. This scheme is sometimes called an implicit-explicit (IMEX) scheme, since we have used an explicit scheme on the slow terms. We note that there is almost nothing[3] stopping us from performing an analogous reduction (called a Schur complement reduction) to the one done for the filtered equations,

$$h^{n+1} - \Delta t^2 \nabla \cdot \left(g h^n \nabla h^{n+1} \right) = h^n - \Delta t \nabla \cdot (h\mathbf{u})^* \,, \qquad (9)$$

representing the centered time-difference formula for the (linearized) implicitly-stepped wave equation $h_{tt} = \nabla \cdot (g h \nabla h)$.

V. Choice of Implicit Time-Stepping Formula

The "2^{nd} Dahlquist Barrier" tells us there is no implicit time-stepping formulas of order > 2 that is A-stable (stability region contains the whole left-half plane), and that the trapezoid rule (TR) has the smallest error coefficient of any second order implicit formula. Higher-order backward differentiation formulas (BDF) are also worthy of consideration.

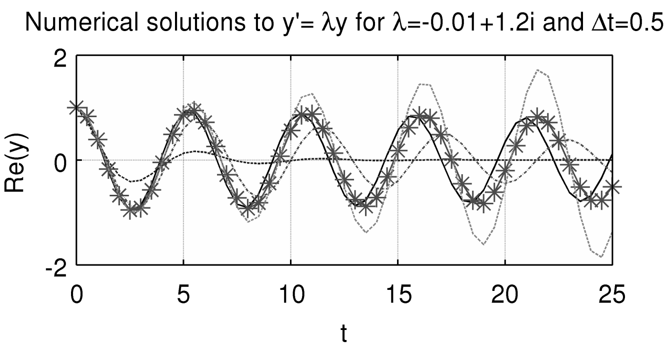

Figure 1: *Comparison of implicit time-steppers for the DE $y' = \lambda y$, $\lambda = -0.01 + 1.2i$. The different curves correspond to: exact solution (solid, black), BE (dashed, black), TR (stars), BDF2 (dash-dots, dark grey), BDF3 (dashed, light grey).*

In Figure 1, we compare a variety of implicit time-steppers for a lightly damped oscillator. We note that BE and BDF2 are overly diffusive, and BDF3 exhibits a peculiar instability. TR shows a small phase error.

VI. A trivial 1D problem

As a toy-problem, consider the (unregularized) nonlinear long-wave equation (NLWE)

$$\eta_t + (c_0 + \eta)\eta_x = 0 \,, \qquad (10)$$

with $c_0 = 5$, subject to the sech-shaped initial condition shown in Figure 2. This equation has been chosen since it is a nonlinear problem for which exact solutions are attainable via the method of characteristics[4].

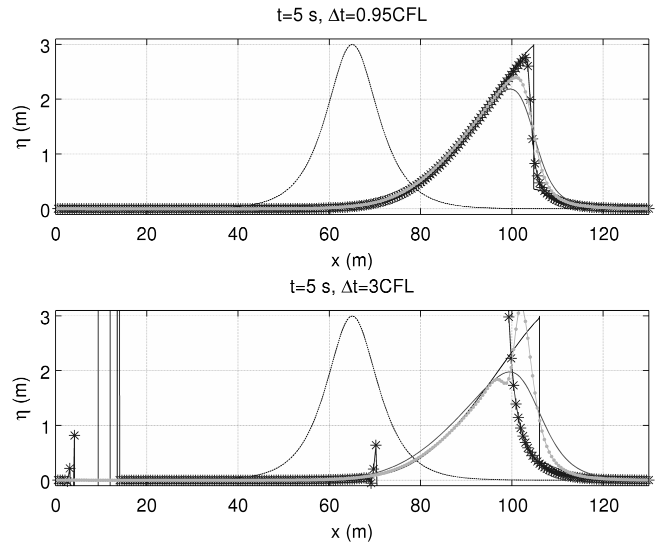

Figure 2: *Comparison of numerical solutions to the NLWE with first-order upwinding and 250 grid points. Top panel: small Δt, bottom: large Δt. The different curves correspond to: IC (dashed), exact solution (solid, black), FE/Godunov (stars, black), TR (dots, light grey), BE (solid, dark grey).*

Method	Δt	Rel. L^2 Error ($t = 5$)	% Mass Loss
FE	small	0.093	4.8%
BE	small	0.25	2.3%
TR	small	0.19	2.9%
FE	big	NaN	NaN
BE	big	0.33	2.0%
TR	big	0.23	4.5%

Table 1: *Relative L^2 error and percentage total mass loss of the runs in Figure 2. If the problem were linear, then all 'Mass Loss' entries would be 0% to numerical precision (except possibly FE/big).*

VII. Conclusions

To accurately propagate waves, satisfying the CFL condition is a must, but if the waves are simply there to conserve mass, then taking large time-steps along with an A-stable implicit stepper (e.g., BE, BDF2, TR) may be an acceptable course of action. It should be noted that the above discussion is not limited to oceanic problems, and it applies to many other equation sets that have both hyperbolic and filtered forms.

In the end, it is the modeller's choice to either keep the fast compressible waves or deal with imposing the incompressibility constraint within the particular spatial discretization method's framework. How easy it is to implement one versus the other will ultimately play a role in arriving at a decision.

References

Books by: Durran (2010); LeVeque (2002, 2007); Cushman-Roisin & Beckers (2011). Paper by: Giraldo *et al.* (2010) in *J. Sci. Comput.*

[3] Keeping it written as a hyperbolic operator may be necessary depending on the spatial discretization method!

[4] At $t \approx 3.4$, the classical solution breaks when the characteristics cross, and the entropy-consistent weak solution (a shock) is recovered by a numerical post-processing technique.