FCT and TVD Algorithms for the 1D Advection Equation

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1 Introduction

When using numerics to simulate hyperbolic PDE's, if not careful, non-physical behaviours may develop. *Flux Corrected Transport* (FCT) algorithms are designed to keep the numerical solution in a more physical regime at the price of doing more work. FCT algorithms are very similar to *Total Variation Diminishing* (TVD) algorithms in the sense that both aim to keep results physical. This can be very valuable in cases when layer depths or densities are class to zero.

2 The Equations

We looked at a simple advection equation with periodic boundary conditions

$$(1) \qquad \frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} = 0, \quad \begin{cases} q(0,t) = q(1,t) \\ q(x,0) = q_0(x) \end{cases}$$

Where c is a constant. (For this paper, taken to be $c = 1$). Recall that the exact solution is simply

$$q(x,t) = q_0(x - ct)$$

3 TVD Scheme

TVD schemes ensure that the total variation does not increase in time. The *TVD* condition:

$$\text{TV}(q) = \sum_{j=1}^{N-1} |q_{j+1} - q_j|$$
$$\text{TV}(q^{n+1}) \le \text{TV}(q^n)$$

It is important to note that TVD schemes are not necessarily positivity-preserving. The advantage of these schemes is that they prevent spurious ripples from appearing near discontinuities.

4 FCT Scheme

We adopted the 6 step algorithm presented by *Boris and Book*. For each timestep:

1. Compute the low order approximation fluxes of $\frac{\partial q}{\partial x}$, denoted Q_x^L

2. Compute the high order approximation fluxes of $\frac{\partial q}{\partial x}$, denoted Q_x^H

3. Define the "antidiffusive fluxes"
$$A = Q_x^H - Q_x^L$$

4. Compute the time advanced low order solution
$$q^{td} = q^n - \Delta t \Delta x^{-1} \left[Q_{right}^L - Q_{left}^L \right]$$

5. Limit the antidiffusive fluxes in a manner such that the next timestep of q will not take on nonphysical values:
$$A_i^C = C_i A_i, \qquad 0 \le C_i \le 1$$

6. Finally, apply the limited antidiffusive fluxes:
$$q^{n+1} = q^{td} - \Delta t \Delta x^{-1} \left[A_{right}^C - A_{left}^C \right]$$

Most of the work of this algorithm is done in step 5 of this method. *Boris and Book* introduced a simple limiter on the antidiffusive fluxes in their 1976 paper. *Zalesak* presented a reworked version of the corrector in the late 70's.

5 Results

We implemented 3 different schemes in MATLAB for this equation: FCT with Boris and Book corrector, Lax-Friedrichs TVD and FCT with Zalesak corrector. We used 50 spatial steps and integrated the solution over one full period.

Our implementations of the Boris and Book corrector yielded poor results; the advection was incredibly dispersive and the accuracy would decrease with smaller time steps. The Lax-Friedrichs TVD scheme was also very dispersive and yielded poor results. However, the FCT algorithm with the Zalesak corrector was much more reasonable and had much better accuracy. The results are shown in figure 1.
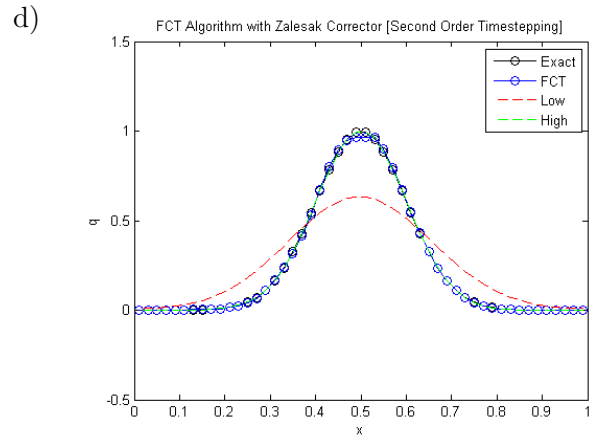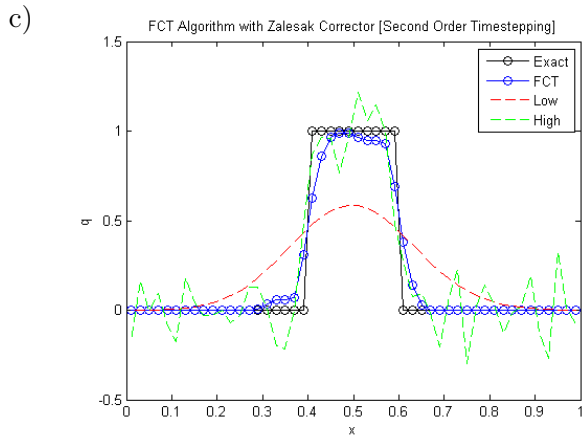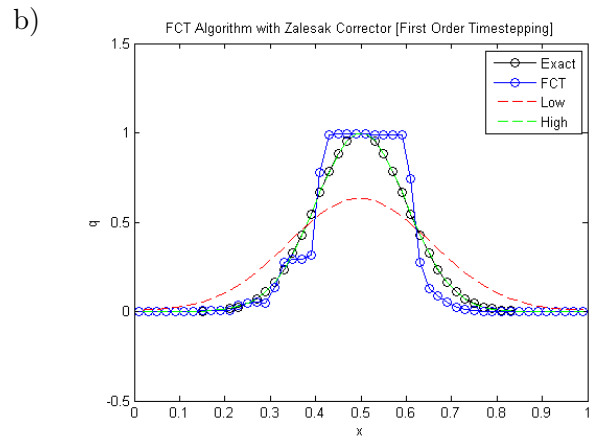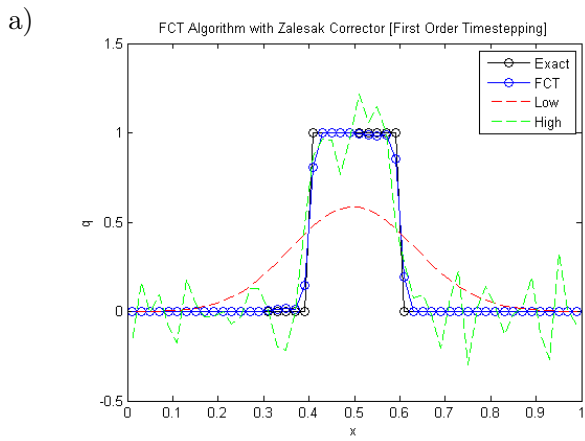
Figure 1: Plots of the FCT algorithm with Zalesak Corrector for a) step function, first order timestepping, b) Gaussian, first order timestepping, c) step function, second order timestepping, d) Gaussian, second order timestepping.