On Energy Budgets and C grids

My eddy-internal wave research lives in a box shaped domain; open at the x boundaries, periodic in y and closed in z with a flat rigid surface and bottom. An eddy is synthesised in the interior of the domain, and internal waves forced at the west boundary propagate inward and soon interact with the eddy. Hydrostatic simulations show that waves of different modes are formed, and the question remains how to quantify the results.

Global energy budget

An energy budget is easily constructed for this situation. It reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_V \rho_0\frac{1}{2}(u^2+v^2)\,dV = -\iiint\limits_V \rho' g w\,dV - \iint\limits_{\delta V} \vec{u}\cdot\hat{n}[p' + \rho_0\frac{1}{2}(u^2+v^2)]\,dS, \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_V \frac{g^2\rho'^2}{2N^2\rho_0}\,dV = +\iiint\limits_V \rho' g w\,dV - \frac{g^2}{2\rho_0 N^2}\iint\limits_{\delta V} \vec{u}\cdot\hat{n}\rho'^2\,dS, \tag{2}$$

where the flux terms can be switched to their divergence form via the divergence theorem,

$$\iint\limits_{\delta V} \vec{u}\cdot\hat{n}[p' + \rho_0\frac{1}{2}(u^2+v^2)]\,dS = \iiint\limits_V \vec{\nabla}\cdot\vec{u}[p'+\rho_0\frac{1}{2}(u^2+v^2)]\,dV, \quad \text{and} \quad \iint\limits_{\delta V}\vec{u}\cdot\hat{n}\rho'^2\,dS = \iiint\limits_V \vec{\nabla}\cdot\vec{u}\rho'^2\,dV. \tag{3}$$

Modal energy budgets

If we substitute into the momentum and density equations the decomposition

$$\{u,v,p'\} = \{u_0,v_0,p'_0\} + \sum_{n=1}^{\infty}\{u_n,v_n,p'_n\}(x,y,t)\cos(m_n z), \qquad \{w,\rho'\} = \sum_{n=1}^{\infty}\{w_n,\rho'_n\}(x,y,t)\sin(m_n z), \tag{4}$$

we can derive modal momentum and density equations. Further applying the usual energy budget derivation procedure, we eventually obtain the modal energy budgets, which read

$$\text{Barotropic KE:}\quad \frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_A \rho_0\frac{H}{2}(u_0^2+v_0^2)\,dA = -H\oint\limits_{\delta A}(\vec{u}_{h_0}\cdot\hat{n})p'_0\,dS - \rho_0 H\iint\limits_A \vec{u}_{h_0}\cdot\vec{N}_0^u\,dA, \tag{5}$$

$$\text{Modal KE:}\quad \frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_A \frac{\rho_0 H}{4}(u_i^2+v_i^2)\,dA = -\frac{H}{2}\iint\limits_A \rho'_i g w_i\,dA - \frac{H}{2}\oint\limits_{\delta A}(\vec{u}_{h_i}\cdot\hat{n})p'_i\,dS - \frac{\rho_0 H}{2}\iint\limits_A \vec{u}_{h_i}\cdot\vec{N}_i^u\,dA, \tag{6}$$

$$\text{Modal APE:}\quad \frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_A \frac{Hg^2}{4\rho_0 N_0^2}\rho_i'^2\,dA = +\frac{H}{2}\iint\limits_A \rho'_i g w_i\,dA - \frac{Hg^2}{2\rho_0 N_0^2}\iint\limits_A \rho'_i N_i^\rho\,dA, \tag{7}$$

Most of the terms are familiar, except for the last term in each equation. Figure 1 shows the first few N_i^u and N_i^ρ.

C grids

Which of these two equations is correct for computing the (horizontal) kinetic energy of the grid box shown in Fig 2?

$$KE = \rho_0\frac{1}{2}\left[\left(\frac{1}{2}u(i,j)^2 + \frac{1}{2}u(i+1,j)^2\right) + \left(\frac{1}{2}v(i,j)^2 + \frac{1}{2}v(i,j+1)^2\right)\right]\Delta x\Delta y \tag{8}$$

$$KE = \rho_0\frac{1}{2}\left[\left(\frac{1}{2}u(i,j) + \frac{1}{2}u(i+1,j)\right)^2 + \left(\frac{1}{2}v(i,j) + \frac{1}{2}v(i,j+1)\right)^2\right]\Delta x\Delta y \tag{9}$$

$$\vec{N}_0^u = (\vec{u}_{h_0} \cdot \vec{\nabla}_h)\vec{u}_{h_0} + \frac{1}{2}\sum_{i=1}^{\infty}\Big[(\vec{u}_{h_i} \cdot \vec{\nabla}_h)\vec{u}_{h_i} - w_i \vec{u}_{h_i} m_i\Big], \tag{10}$$

$$\vec{N}_1^u = (\vec{u}_{h_0} \cdot \vec{\nabla}_h)\vec{u}_{h_1} + (\vec{u}_{h_1} \cdot \vec{\nabla}_h)\vec{u}_{h_0} + \frac{1}{2}\sum_{i=1}^{\infty}\Big[(\vec{u}_{h_i} \cdot \vec{\nabla}_h)\vec{u}_{h_{i+1}} + (\vec{u}_{h_{i+1}} \cdot \vec{\nabla}_h)\vec{u}_{h_i} - w_i \vec{u}_{h_{i+1}} m_{i+1} + w_{i+1}\vec{u}_{h_i} m_i\Big], \tag{11}$$

$$\begin{aligned}\vec{N}_2^u = {} & (\vec{u}_{h_0} \cdot \vec{\nabla}_h)\vec{u}_{h_2} + (\vec{u}_{h_2} \cdot \vec{\nabla}_h)\vec{u}_{h_0} + \frac{1}{2}\Big[(\vec{u}_{h_1} \cdot \vec{\nabla}_h)\vec{u}_{h_1} + w_1 \vec{u}_{h_1} m_1\Big] \\ & + \frac{1}{2}\sum_{i=1}^{\infty}\Big[(\vec{u}_{h_i} \cdot \vec{\nabla}_h)\vec{u}_{h_{i+2}} + (\vec{u}_{h_{i+2}} \cdot \vec{\nabla}_h)\vec{u}_{h_i} - w_i \vec{u}_{h_{i+2}} m_{i+2} + w_{i+2}\vec{u}_{h_i} m_i\Big],\end{aligned} \tag{12}$$

$$\begin{aligned}\vec{N}_3^u = {} & (\vec{u}_{h_0} \cdot \vec{\nabla}_h)\vec{u}_{h_3} + (\vec{u}_{h_3} \cdot \vec{\nabla}_h)\vec{u}_{h_0} + \frac{1}{2}\Big[(\vec{u}_{h_1} \cdot \vec{\nabla}_h)\vec{u}_{h_2} + (\vec{u}_{h_2} \cdot \vec{\nabla}_h)\vec{u}_{h_1} + w_2 \vec{u}_{h_1} m_1 + w_1 \vec{u}_{h_2} m_2\Big] \\ & + \frac{1}{2}\sum_{i=1}^{\infty}\Big[(\vec{u}_{h_i} \cdot \vec{\nabla}_h)\vec{u}_{h_{i+3}} + (\vec{u}_{h_{i+3}} \cdot \vec{\nabla}_h)\vec{u}_{h_i} - w_i \vec{u}_{h_{i+3}} m_{i+3} + w_{i+3}\vec{u}_{h_i} m_i\Big],\end{aligned} \tag{13}$$

$$N_1^\rho = \Big[(\vec{u}_{h_0} \cdot \vec{\nabla}_h)(\rho_1') + \frac{1}{2}\sum_{i=1}^{\infty}\Big((\vec{u}_{h_i} \cdot \vec{\nabla}_h)\rho_{i+1}' - (\vec{u}_{h_{i+1}} \cdot \vec{\nabla}_h)\rho_i'\Big)\Big] + \frac{1}{2}\sum_{i=1}^{\infty}\Big(w_{i+1}\rho_i' m_i - w_i \rho_{i+1}' m_{i+1}\Big), \tag{14}$$

$$\begin{aligned}N_2^\rho = {} & \Big[(\vec{u}_{h_0} \cdot \vec{\nabla}_h)(\rho_2') + \frac{1}{2}\sum_{i=1}^{\infty}\Big((\vec{u}_{h_i} \cdot \vec{\nabla}_h)\rho_{i+2}' - (\vec{u}_{h_{i+2}} \cdot \vec{\nabla}_h)\rho_i'\Big) + \frac{1}{2}(\vec{u}_{h_1} \cdot \vec{\nabla}_h)\rho_1'\Big] \\ & + \frac{1}{2}\Big[w_1 \rho_1' m_1 + \sum_{i=1}^{\infty}\Big(w_{i+2}\rho_i' m_i - w_i \rho_{i+2}' m_{i+2}\Big)\Big],\end{aligned} \tag{15}$$

$$\begin{aligned}N_3^\rho = {} & \Big[(\vec{u}_{h_0} \cdot \vec{\nabla}_h)(\rho_3') + \frac{1}{2}\sum_{i=1}^{\infty}\Big((\vec{u}_{h_i} \cdot \vec{\nabla}_h)\rho_{i+3}' - (\vec{u}_{h_{i+3}} \cdot \vec{\nabla}_h)\rho_i'\Big) + \frac{1}{2}\Big((\vec{u}_{h_1} \cdot \vec{\nabla}_h)\rho_2' + (\vec{u}_{h_2} \cdot \vec{\nabla}_h)\rho_1'\Big)\Big] \\ & + \frac{1}{2}\Big[w_2 \rho_1' m_1 + w_1 \rho_2' m_2 + \sum_{i=1}^{\infty}\Big(w_{i+3}\rho_i' m_i - w_i \rho_{i+3}' m_{i+3}\Big)\Big].\end{aligned} \tag{16}$$

Figure 1: The nonlinear terms up to the third mode.

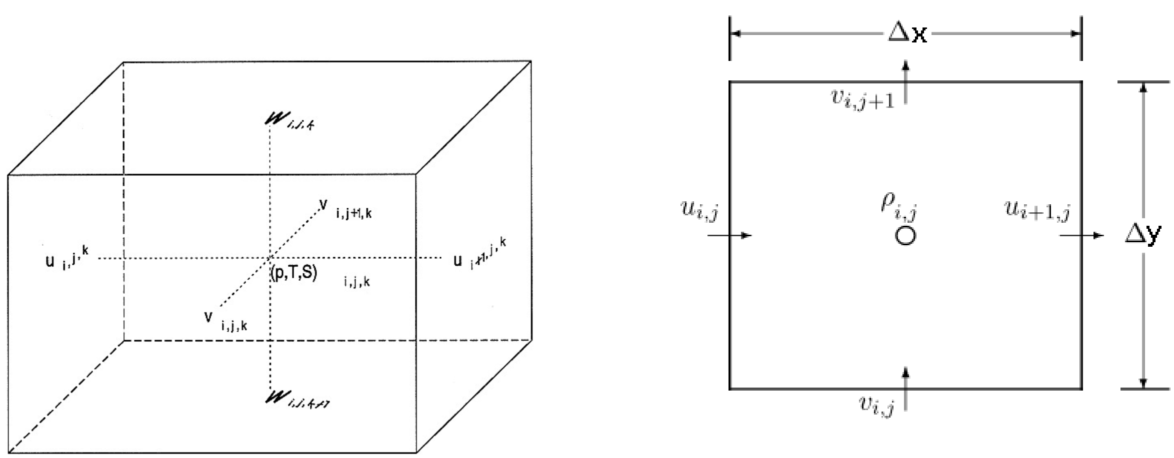

Figure 2: The Arakawa C-grid staggering of variables, used by the MITgcm. Left: 3D view, Right: top view.